Quadratics

A quadratic equation is one of the form $ax^2 + bx + c = 0$.
A quadratic function is one of the form $f(x) = ax^2 + bx + c$

Solving Quadratic Equations

We can solve a quadratic equation in four ways

- (i) by factorising
- (ii) by completing the square
- (iii) by using the formula
- (iv) graphically

Let's look at each of these in more detail.

Method (i) **Factorising**
Type 1 where $a = 1$

e.g. $x^2 + 5x + 6 = 0$

We need to find two numbers that multiply to 6 and add to give 5.
They are 2 and 3.
Then we rewrite the equation as

$$\underline{x^2 + 2x}\underline{\underline{+3x + 6}} = 0 \quad \text{and then we factorise the two parts}$$

$$\underline{x(x+2)}\underline{\underline{+3(x+2)}} = 0 \quad \text{and then we factorise that.}$$

$$(x+2)(x+3) = 0$$

if two things multiply to give 0 then one of them must be 0, so

$(x+2) = 0$ or $(x+3) = 0$
i.e. $x = -2$ or $x = -3$

We have to be careful if there are minus signs

e.g.. $x^2 - 4x - 12 = 0$

We need to find two numbers that multiply to -12 and add to give -4.
They are 2 and -6.
Then we rewrite the equation as

$$\underline{x^2 + 2x}\underline{\underline{-6x - 12}} = 0$$

$$\underline{x(x+2)}\underline{\underline{-6(x+2)}} = 0 \quad \text{remember that } (-6)\times(+2) = -12$$

$$(x+2)(x-6) = 0$$

So $(x+2) = 0$ or $(x-6) = 0$
i.e. $x = -2$ or $x = 6$

Type 2 where $a \neq 1$

e.g. $2x^2 - 5x - 3 = 0$

We add in an extra step we multiply the 2 by the -3 first to get -6.
Then we need two numbers that multiply to give -6 and add to give -5.
They are -6 and 1.
Then we rewrite the equation as.

$$2x^2 - 6x + x - 3 = 0$$
$$2x(x-3) + 1(x-3) = 0$$
$$(x-3)(2x+1) = 0$$

So $(x-3) = 0$ or $(2x+1) = 0$

i.e. $x = 3$ or $x = -1/2$

Method (ii) Completing the square

e.g. $x^2 + 6x + 5 = 0$

Rewrite this as $(x+3)^2 - 9 + 5 = 0$ since $(x+3)^2 = (x+3)(x+3) = x^2 + 6x + 9$

So $(x+3)^2 = 4$

So $(x+3) = \pm\sqrt{4} = \pm 2$

So $x + 3 = 2$ or $x + 3 = -2$

So $x = -1$ or $x = -5$

e.g. 2 $2x^2 - 5x - 3 = 0$

First we take out a factor of 2, because this is an equation we can divide everything by 2, if it was a function we would have to leave the factor outside the bracket.

So we get $x^2 - 5/2\,x - 3/2 = 0$

So $(x - 5/4)^2 - 25/16 - 3/2 = 0$

$(x - 5/4)^2 = 49/16$

$(x - 5/4) = \pm\sqrt{49/16} = \pm 7/4$

$x - 5/4 = 7/4$ or $x - 5/4 = -7/4$

$x = 3$ or $x = -1/2$

If we were completing the square with the function we would get

$$y = 2x^2 - 5x - 3$$
$$y = 2(x^2 - 5/2\,x - 3/2)$$
$$y = 2[(x - 5/4)^2 - 25/16 - 3/2]$$
$$y = 2[(x - 5/4)^2 - 49/16]$$
$$y = 2(x - 5/4)^2 - 49/8$$

Method (iii) Using the formula

The formula is $$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

e.g. $2x^2 - 5x - 3 = 0$

so $$x = \frac{-(-5) \pm \sqrt{(-5)^2 - 4(2)(-3)}}{2(2)}$$

$$x = \frac{5 \pm \sqrt{25 - (-24)}}{4}$$

$$x = \frac{5 \pm \sqrt{49}}{4}$$

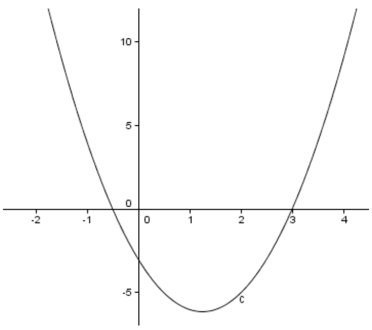
$$x = \frac{5 \pm 7}{4}$$

$$x = \frac{12}{4} = 3 \text{ or } x = \frac{-2}{4} = \frac{-1}{2}$$

Method (iv) Graphically

Plot the graph of the equation and see where it crosses the *x* axis.

e.g to solve $2x^2 - 5x - 3 = 0$ we plot the graph of $y = 2x^2 - 5x - 3$



We can see the graph crosses the *x* axis at -0.5 and 3, so the solutions to the equation are *x* = -0.5 and *x* = 3.

The number of solutions to a quadratic equation.

A quadratic equation can have

(i) no real solutions
(ii) exactly 1 real solution
(iii) exactly two real solutions.

We can use part of the formula to determine this.

The determinant or discriminant of a quadratic is given by $\Delta = \sqrt{b^2 - 4ac}$

If $\Delta < 0$ then the equation has no real solutions.

If $\Delta = 0$ then the equation has exactly one real solutions.

If $\Delta > 0$ then the equation has exactly two real solutions.

e.g. How many solutions does $2x^2 + 4x + 3 = 0$ have?

$\Delta = b^2 - 4ac = 4^2 - 4(2)(3) = 16 - 24 = -8$ so $\Delta < 0$ so there are no real solutions.

e.g. How many solution does $9x^2 + 6x + 1 = 0$

$\Delta = b^2 - 4ac = 6^2 - 4(9)(1) = 36 - 36 = 0$ so $\Delta = 0$ so there is exactly one real solution.

e.g. How many solution does $3x^2 + 2x - 4 = 0$

$\Delta = b^2 - 4ac = 2^2 - 4(3)(-4) = 4 - (-48) = 52$ so $\Delta > 0$ so there are exactly two real solutions.

Graphing Quadratic Functions

There are four important points on a quadratic graph

(i) the turning point
(ii) the y intercept
(iii) the two x intercepts

The turning point can either be a maximum or a minimum.

If a > 0 the graph will look like  and have a minimum point.

If a < 0 the graph will look like  and have a maximum point

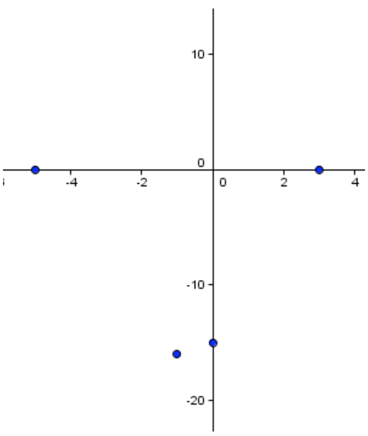
To find these points we need the three forms of the quadratic.

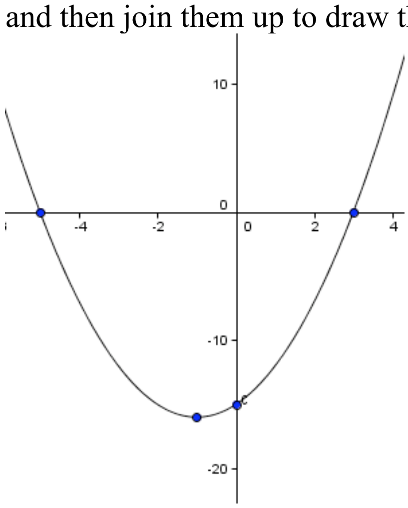
e.g.	$y = x^2 + 2x - 15$
the factorised form is	$y = (x-3)(x+5)$
the completing the square form is	$y = (x+1)^2 - 16$

We find

(i) the turning point from $y = (x+1)^2 - 16$ the turning point is (-1,-16) and it is a minimum
(ii) the y intercept from $y = x^2 + 2x - 15$ the y intercept is (0,-15)
(iii) the two x intercepts from $y = (x-3)(x+5)$ the x intercepts are (3,0) and (-5,0)

We plot these points and then join them up to draw the graph.





e.g.2 $y = -2x^2 + 5x + 3$

the factorised form is $y = -(2x+1)(x-3)$

the completing the square form is $y = -2(x - \frac{5}{4})^2 + \frac{49}{8}$

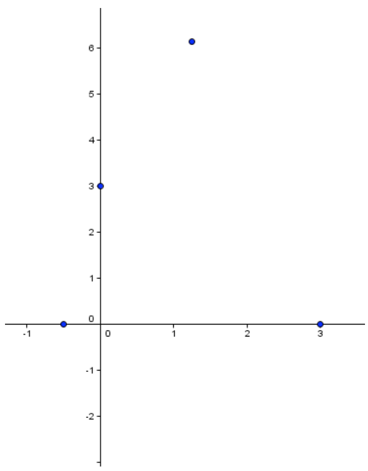
We find

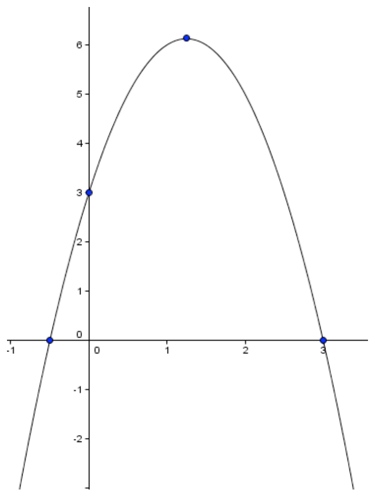
(i) the turning point from $y = -2(x - \frac{5}{4})^2 + \frac{49}{8}$ the turning point is $(\frac{5}{4}, \frac{49}{8})$ and it is a mamimum.

(ii) the y intercept from $y = -2x^2 + 5x + 3$ the y intercept is (0,3)

(iii) the two x intercepts from $y = -(2x+1)(x-3)$ the x intercepts are $(\frac{-1}{2}, 0)$ and (3,0)

We plot these points and then join them up to draw the graph.



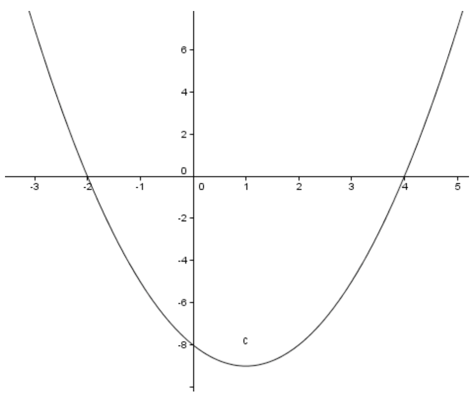


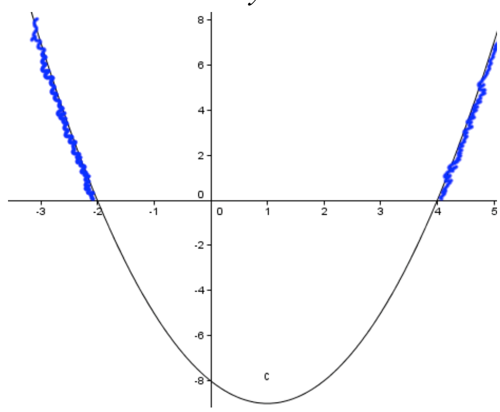
Solving quadratic inequalities

We can use a graph of a quadratic function to solve a quadratic inequality.

e.g. Solve $x^2 - 2x - 8 \geq 0$

First we plot the graph of $y = x^2 - 2x - 8$ then on the graph we decide where $x^2 - 2x - 8 \geq 0$ i.e. where $y \geq 0$





From the graph we can see $x^2 - 2x - 8 \geq 0$ when $x \leq -3$ and $x \geq 4$